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Computing minimal finite free resolutions

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Abstract

In this paper we address the basic problem of computing minimal finite free resolutions of homogeneous submodules of graded free modules over polynomial rings. We develop a strategy, which keeps the resolution `minimal at every step`. Among the relevant benefits is a marked saving of time, as the first reported experiments in CoCoA show. The algorithm has been optimized using a variety of techniques, such as minimizing the number of critical pairs and employing an "ad hoc" Hilbert-driven strategy. The algorithm can also take advantage of various a priori pieces of information, such as the knowledge of the Castelnuovo regularity. © 1997 Elsevier Science B.V.

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1. Introduction

In this paper we address the basic problem of computing minimal finite free resolutions (MFFR) of homogeneous submodules of graded free modules over polynomial rings. The theory of finite free resolutions has a long history; it goes back to the fundamental paper by Hilbert [11] and is a cornerstone in Algebraic Geometry and Commutative Algebra, since many geometric invariants can be calculated from such resolutions. Despite their outstanding importance, it was only in the 1980s that the first attempt to actually compute them was tried. For instance [14] considered some computational problems involved, but only for very special ideals. Then in [13] the authors discussed a new computational approach to the problem, which worked in general. The first implementations of the MFFR-algorithms, done in Macaulay 3.0 [3], and CoCoA 1.5 [9] were unsatisfactory for various reasons. Recently we have seen new ideas

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introduced to deal with this problem, which lead to more efficient implementations (see [8]). Other researchers are working on the subject, in particular we are aware of some work in progress by La Scala [12], whose algorithm has been recently implemented in Macaulay 2, and by Caboara-Traverso [4].

Our idea can be briefly described by saying that our MFFR algorithm features a strategy, which has a great deal of flexibility and keeps the minimality at every step. More specifically, our algorithm is a combination of the following ingredients:
- the use of a minimal set of critical pairs, as explained in Section 3;
- the use of left-induced term-orders (see Section 4), as suggested many years ago by Schreyer ([15]);
- the use of Hilbert functions to discard *Type* S_0 critical pairs (see Section 5);
- a "top first" strategy, which first computes the top part of every computational cell, i.e., the part which manages the "critical pairs" and then the bottom part, i.e., the part which manages the "incoming vectors";
- the possibility of placing a priori bounds in the "resolution matrix", when some extra information, such as the Castelnuovo regularity, is known.

The default uses a "reading pattern strategy", i.e., a strategy which computes in increasing degree and from left to right. However the flexibility is such that completely different strategies can be used so that the algorithm may benefit from other sources of knowledge.

The paper is structured in the following way. After having discussed some more or less well known facts in Section 2, we address the problem of minimizing the set of the critical pairs (see Section 3). Then we propose an algorithm to compute a minimal set of syzygies of a graded module (see Section 4). The algorithm embodies a subroutine which uses Hilbert–Poincaré series as a tool for discarding useless critical pairs (see Section 5), and one should be aware of the fact that useless critical pairs are not the pairs which reduce to zero in Buchberger's algorithm, but pairs which, by reducing to zero, produce a syzygy, which reduces to zero. Then Section 6 is devoted to collecting all the ideas coming from the previous parts and describing how to compute in the "computational zone" of the so called "resolution matrix". All the algorithms described in the paper were implemented in CoCoAL, the programming language of CoCoA, before being coded in the kernel of CoCoA 3 [5]. A set of examples, almost all of which come from specific problems in Algebraic Geometry, is the basis for our table of timings. It is described in Section 7, and shows the excellent behavior of our algorithm, which, for instance, performs with many orders of magnitude better than the old CoCoA.

2. Some preparatory results

Let us start by considering the following situation. Let $R := k[X_1, \ldots, X_n]$ be the polynomial ring over a field, graded over $\mathbb{N}$ by $\deg(X_i) > 0$ for $i = 1, \ldots, n$ and let L denote a finitely generated graded free R-module. Then $L \cong \bigoplus_{i=1}^{r} R(-a_i)$, where the a_i's are the degrees of a set of homogeneous generators of L.

Let σ be a term-order on the set of power products of L, i.e., elements of the form $(0,\dots,0,t,0,\dots,0)$, where t is a power product in R.

If M is a submodule of L, we denote by $\mathrm{LT}_\sigma(M)$ the leading term module of M, i.e., the submodule of L generated by $\{\mathrm{LT}_\sigma(m) \mid m \in M, m \neq 0\}$.

If M is an $\mathbb{N}$-graded submodule of L, then $M = \bigoplus_{d\in\mathbb{N}} M_d$ and the elements of M_d are called homogeneous elements of degree d of M; we make the following:

Convention 1. Homogeneous generators of a graded module are written in nondecreasing order with respect to their *degree*.

In practice it is clear that the generators, however given, can be rearranged to meet the requirements of Convention 1.

Suppose now that we want to compute a Gröbner basis of M by means of Buchberger's algorithm.

Definition 1. A strategy which computes a Gröbner basis in nondecreasing degree is called a degree-compatible strategy. A strategy which always makes full reduction, i.e., it stops the reduction of every element when no power product in the support can be reduced, is called a fully-reductive strategy.

Convention 2. We assume that we always use degree-compatible fully reductive strategies.

While the use of a degree-compatible strategy is essential in the following, the use of a fully reductive strategy is only optional.

Definition 2. We use the expression "d-truncated Gröbner basis" of M to mean the subset of all the elements of a Gröbner basis of M having degree not greater than d and we denote it by $G_{\leq d}(M)$. We call $M_{\leq d}$ the module generated by the elements of M of degree not greater than d.

It is very well known that if M is a graded module and a degree-compatible strategy is used to compute a Gröbner basis of M, then, after passing a degree d, a d-truncated Gröbner basis of M is already computed.

Lemma 3. *Let M be a graded submodule of a graded free module L and let $G_{\leq d}(M)$ be a d-truncated Gröbner basis of M. Let $\{n_1,\dots,n_s\}$ be homogeneous elements of L of degree d, such that the leading terms of the n_i's are*

(a) *pairwise different;*

(b) *not divided by any leading terms of the elements of $G_{\leq d}(M)$.*

Let $M' := M + (n_1,\dots,n_s)$. Then $G_{\leq d}(M) \cup \{n_1,\dots,n_s\}$ is a d-truncated Gröbner basis of M'.

Proof. A simple analysis of the possible critical pairs leads to the conclusion. □

Proposition 4. *Let $M, M', n_1, \ldots, n_s$ be as in Lemma 3; assume that M is minimally generated by $\{m_1, \ldots, m_r\}$. Then M' is minimally generated by $\{m_1, \ldots, m_r\} \cup \{n_1, \ldots, n_s\}$.*

Proof. The only possibility against minimality is that there is a linear combination $\sum a_i n_i + \sum b_j m_j = 0$, with $a_i \in k$ for $i = 1, \ldots, s$ and some of the a_i's different from 0. But then the leading term of $m := \sum a_i n_i$ is the leading term of one of the n_i's, hence is not divided by any of the leading terms of elements of $G_{\leq d}(M)$. At the same time m can be written as $-\sum b_j m_j$, hence it reduces to 0 via $G_{\leq d}(M)$. This is a contradiction. □

3. Computing a minimal set of critical pairs

Our next goal is to show that the results of the previous section can also be used to obtain a *minimal* set of critical pairs.

Let $\{m_1, \ldots, m_r\}$ be a set of homogeneous elements of a graded free module $L_0 \cong \bigoplus_{i=1}^{s} R(-a_i)$ and let $M := (m_1, \ldots, m_r)$. If $\{e_1, \ldots, e_s\}$ denotes the canonical basis of L_0, then $\deg(e_i) = a_i$. We recall that the power products of L_0 are the elements of the form $t \cdot e_i$, where $t = X_1^{p_1} \cdots X_n^{p_n}$ and we explicitly remark that $\deg(t \cdot e_i) = \deg(t) + a_i$.

Suppose we want to compute a Gröbner basis of M with respect to a degree-compatible term-order σ on L_0. We have to run Buchberger's algorithm, so we are interested in the so called *critical pairs*.

We recall that two elements n_i and n_j of L_0 give rise to a critical pair if $\mathrm{LT}(n_i)$ and $\mathrm{LT}(n_j)$ are of type $s \cdot e_l$ and $t \cdot e_l$, with the same l. We introduce new labels ε_i's to indicate the position, so that the critical pair can be described by

$$-\frac{\mathrm{lcm}(s,t)}{s}\varepsilon_i + \frac{\mathrm{lcm}(s,t)}{t}\varepsilon_j.$$

We simply refer to it as $P(i,j)$ and we observe that its degree is $\deg(P(i,j)) := \deg(\mathrm{lcm}(s,t)) + \deg(e_l)$. In this way we embody the critical pairs in a graded module, which we call $\mathcal{M}$.

Given a critical pair $-a\varepsilon_i + b\varepsilon_j$, of course $\mathrm{LT}(a \cdot n_i) = \mathrm{LT}(b \cdot n_j)$, so we need a convention to break the tie.

Convention 3. Let $i < j$ and $P(i,j) := -a\varepsilon_i + b\varepsilon_j$, we put

$$\mathrm{LT}(P(i,j)) := b\varepsilon_j.$$

Corollary 5. *With the above construction and convention, we have a term-order on the R-module $\mathcal{M}$ generated by the critical pairs.*

To process the critical pair $P(i,j)$ means to compute the normal form of $-(\mathrm{lcm}(s,t)/s)n_i + (\mathrm{lcm}(s,t)/t)n_j$, where the normal form is taken with respect to the part of the Gröbner basis already computed. We observe that such a normal form is a homogeneous element of degree $\deg(P(i,j))$. We also observe that such normal forms depend on the order chosen to process the critical pairs. This suggest the following:

Convention 4. We always use the following order on the critical pairs of a given degree: $P(i,j) < P(h,k)$ if *either* $j < k$ *or* $j = k$ and $i < h$.

For every new element n_k added to the Gröbner basis, we need to add a new label ε_k, so that the new pairs $P(i,k)$ can be written in the form $-a\varepsilon_i + b\varepsilon_k$. This means that for every new element n_k added to the Gröbner basis, we upgrade the module containing the critical pairs and we obtain a new module $\mathscr{M}_k$. We observe that $\mathscr{M} = \bigcup_k \mathscr{M}_k$. This does not cause any problem to the algorithm which computes a Gröbner basis of the critical pairs, since each computation among the currently given pairs keeps inside the module $\mathscr{M}_k$. Therefore we can use the results of the previous section to produce a minimal set of generators of the critical pairs. As we have already noted, two elements of the critical pairs produce a critical pair (of critical pairs) only if they are of type $P(i,k)$, $P(j,k)$ (same k).

Now we are ready to state the main result.

Theorem 6. *Let i,j,k be three indices, such that $i < j < k$. Let $P(i,k) := -a\,\varepsilon_i + c_i\,\varepsilon_k$, $P(j,k) := -b\,\varepsilon_j + c_j\,\varepsilon_k$, $P(i,j) := -A\,\varepsilon_i + B\,\varepsilon_j$. There are three possibilities.*

(1) If $c_i \mid c_j$ then $P(j,k)$ can be deleted from the minimal set of pairs.

(2) If c_j properly divides c_i then $P(i,k)$ can be deleted from the minimal set of pairs.

(3) No divisibility occurs. Then $P(i,j)$ can be deleted from the minimal set of pairs if and only if $\gcd(a,b) = 1$.

Proof. In the first case we have $P(i,k) < P(j,k)$. Let $c_j = d \cdot c_i$. Then $P(j,k) = d \cdot P(i,k) + \alpha \cdot P(i,j)$. Therefore not only $P(j,k)$ can be deleted from the minimal set of pairs, but also it reduces to 0, hence it does not contribute to the Gröbner basis of the pairs.

In the second case let $c_i = e \cdot c_j$ we have $P(i,k) = e \cdot P(j,k) + \alpha \cdot P(i,j)$. We observe that $\deg(P(i,k)) > \deg(P(j,k))$ and $P(i,k)$ reduces to 0, hence it can be deleted from the minimal set of pairs; moreover it does not contribute to the Gröbner basis of the pairs.

In the third case we let $C := \operatorname{lcm}(c_i, c_j)$ and we see that the critical pair associated to $(P(i,k), P(j,k))$ produces $-(C/c_i)a\varepsilon_i + (C/c_j)b\varepsilon_j$, which is therefore a multiple of $P(i,j)$, i.e., $((C/c_i)a\varepsilon_i + (C/c_j)b\varepsilon_j) = R \cdot (-A\varepsilon_i + B\varepsilon_j)$. If $\gcd(a,b) \neq 1$, then the produced pair $-(C/c_i)a\varepsilon_i + (C/c_j)b\varepsilon_j$ is a strict multiple of $P(i,j)$, hence it reduces to 0 and does not contribute either to the minimal set of pairs or to their Gröbner basis. If $\gcd(a,b) = 1$, then the produced pair $-(C/c_i)a\varepsilon_i + (C/c_j)b\varepsilon_j$ is equal to $P(i,j)$, hence the pair $P(i,j)$ is taken out from the minimal set of pairs. □

Remark 7. From the preceding theorem we derive a procedure, which we call `New_Pairs(K)` and which builds the new minimal pairs, when a new element is added to the Gröbner basis, and removes the nonminimal ones from the pairs to be processed. If the new element is the Kth one, then the computational cost of `New_Pairs(K)` is $\mathcal{O}(K)$ divisibility and $\mathcal{O}(K^2)$ coprimality tests between power products. Both tests can be efficiently performed if the power products in the components of the critical pairs keep the information of their "square-free part", as described in [2].

Remark 8. The Procedure `New_Pairs(K)` can be interpreted as an improved version of the Gebauer–Möller [7] criteria in the homogeneous case.

4. Computing minimal syzygies

Let $\{m_1,\dots,m_r\}$ be a set of homogeneous elements of L_0, with $d_i := \deg(m_i)$, $i=1,\dots,r$; $d_1 \le \cdots \le d_r$, let $M := (m_1,\dots,m_r)$, $L_1 := \bigoplus_{i=1}^r R(-d_i)$ and $\{\varepsilon_1,\dots,\varepsilon_r\}$ the canonical basis of L_1. Then M can be represented as the image of the map of graded free modules

$$\Phi : L_1 \to L_0 \quad \text{where } \Phi(\varepsilon_i) = m_i;\ \ i=1,\dots,r.$$

We observe that, because of the chosen shifts, Φ turns out to be a graded homomorphism.

Now we consider a term-order σ on L_0; it induces a filtration on L_0. Classical results in the theory of associated graded objects to filtrations (see for instance [15, 14, 6]) suggest the following.

Definition 9. A term-order τ on L_1 is said to be induced by σ and Φ if the following condition holds: for every power products s and t, then

$$t\cdot\varepsilon_i >_\tau s\cdot\varepsilon_j \quad \text{if } \mathrm{LT}_\sigma(tm_i) >_\sigma \mathrm{LT}_\sigma(sm_j).$$

A term-order τ on L_1 is said to be the term-order left-induced by σ and Φ if it is defined by

$$t\cdot\varepsilon_i >_\tau s\cdot\varepsilon_j \quad \text{if}$$

$$\text{either} \quad \mathrm{LT}_\sigma(tm_i) >_\sigma \mathrm{LT}_\sigma(sm_j) \quad \text{or} \quad \mathrm{LT}_\sigma(tm_i) = \mathrm{LT}_\sigma(sm_j) \quad \text{and} \quad i > j.$$

Convention 5. Henceforth we will use the term-order left-induced by σ and Φ.

Remark 10. The use of left-induced term-orders τ was already suggested by Schreyer [15] in his Diploma thesis. It is certainly not an original contribution of this paper; however we want to remark that it is a "natural" choice. Namely, it is the correct choice to make the morphism Φ compatible with the Gröbner filtrations induced on L_0 and L_1 by σ and τ, respectively.

Definition 11. With the above notations and conventions, minimal critical pairs are of two different types, namely:

Type G if the corresponding normal form is different from zero, hence it yields a new element in the Gröbner basis of M;

Type S if the corresponding normal form is zero, hence it yields a syzygy of the given set of generators of M.

It is well known (see for instance [1]) that *Type S* critical pairs yield generators of the module of syzygies of $\{m_1, \dots, m_r\}$, but we get more.

Definition 12. Let M be minimally generated by $\{m_1, \dots, m_r\}$. Let a procedure be given which yields syzygies of the m_i's from the *Type S* critical pairs selected among a minimal set of critical pairs as described in Section 3 and processed in the order stated with Convention 4, while computing a Gröbner basis of M; we denote by S_d the set of the corresponding degree-d syzygies. Let G_d be a d-truncated Gröbner basis of the module generated by $\mathrm{Syz}(M_{\leq(d-1)})$. Then a *Type S* critical pair can be of two different types, namely:

Type $S_{\min}$ if it yields a minimal generator for the module of syzygies;
Type S_0 otherwise.

If we apply the results of Section 2 to the above described situation, we get the following:

Proposition 13. *Let M be a graded submodule of a graded free module L_0, assume that M is minimally generated by $\{m_1, \dots, m_r\}$. Then the Type $S_{\min}$ critical pairs described in Definition* 12 *yield a minimal set of generators of* $\mathrm{Syz}(m_1, \dots, m_r)$.

Proof. The proof follows from Proposition 4. □

Corollary 14. *Let M be a graded submodule of a graded free module L_0, assume that M is minimally generated by $\{m_1, \dots, m_r\}$. Then Proposition* 13 *yields an algorithm, which computes the minimal syzygies of the given minimal set of generators* $\{m_1, \dots, m_r\}$.

Corollary 15. *Let M be a graded submodule of a graded free module L_0, assume that M is generated by $\{m_1, \dots, m_r\}$. Then a combination of Propositions* 4 *and* 13 *yields an algorithm, which we call* `MinSyz` *and which computes the minimal syzygies of a minimal subset of the given generators.*

Remark 16. This approach can be implemented by using a "horizontal" strategy, which does the computation degree by degree in both modules. The horizontal approach can be improved if we normalize a syzygy as soon as it is found. To make this remark effective, we must "go from left to right" in the sense that we have to process the pairs of the syzygies in degree d before normalizing the new incoming syzygies in the same degree.

5. The use of the Hilbert functions

In this section we show how to use information coming from numerical functions to possibly detect, hence discard, *Type* S_0 critical pairs.

Definition 17. Let N be a graded submodule of a free graded module over $k[x_1,\dots,x_n]$. We express the Hilbert–Poincaré series of N as $\mathscr{P}_N = \frac{\langle N\rangle}{(1-\lambda)^n}$ and we call $\langle N\rangle$ the "numerator" of N. We denote by $\langle N_{<d}\rangle$ the numerator of the module generated by $N_{<d}$ and we denote by $\langle N_{<d}\rangle^-$ the numerator computed via a $(d-1)$-truncated Gröbner basis of $N_{<d}$. If $P(\lambda)$ is a series or a polynomial, we denote by $\mathrm{cf}_d(P)$ the coefficient of λ^d in $P(\lambda)$. Finally we denote by $\#(S_{\min}(N))_d$ the number of *Type* $S_{\min}$ critical pairs of degree d of N and by $\#(G(N))_d$ the number of *Type G* critical pairs of degree d of N.

When the algorithm `MinSyz` enters a degree d, it computes the d-truncated Gröbner basis of $\mathrm{Syz}(M)_{<d}$ and the d-truncated Gröbner basis of M. At that stage, a minimal set of generators of M of degree not greater than $d-1$ is already computed, say $m_1,\dots,m_r$ of degrees $d_1,\dots,d_r$. We denote by L the free graded module $\bigoplus_i R(-d_i)$. Now it depends on the strategy whether one computes a d-truncated G-basis of $\mathrm{Syz}(M)$ first and then a d-truncated G-basis of M or vice versa. Our default is the first choice, as explained in Remark 16 and in the next section.

Theorem 18. *With the preceding notation, the following equalities hold*:

(i) $\mathrm{cf}_d(\langle M_{<d}\rangle) = \mathrm{cf}_d(\langle M_{<d}\rangle^-) + \#(G(M))_d$;

(ii) $\mathrm{cf}_d(\langle \mathrm{Syz}(M)_{<d}\rangle) = \mathrm{cf}_d(\langle \mathrm{Syz}(M)_{<d}\rangle^-) + \#G(\mathrm{Syz}(M))_d$;

(iii) $\mathrm{cf}_d(\langle \mathrm{Syz}(M)\rangle) = \mathrm{cf}_d(\langle \mathrm{Syz}(M)_{<d}\rangle) + \#(S_{\min}(M))_d$;

(iv) $-\mathrm{cf}_d(\langle \mathrm{Syz}(M)_{<d}\rangle^-) - \mathrm{cf}_d(\langle M_{<d}\rangle^-)$
$= \#G((\mathrm{Syz}(M))_d + \#(G(M))_d + \#(S_{\min}(M))_d$.

Proof. We consider the following sequences

$$0 \to (\mathrm{Syz}(M))_i \to L_i \to (M_{<d})_i \to 0$$

They are clearly exact for $i=0,\dots,d-1$. But we know that $M_{<d}$ is presented through a minimal set of generators, hence the syzygies of M in degree d do not tie elements of M of degree d. Therefore also the dth sequence is exact and the additivity of the Hilbert–Poincaré series implies that

$$\mathrm{cf}_i(\mathscr{P}_L) - \mathrm{cf}_i(\mathscr{P}_{M_{<d}}) - \mathrm{cf}_i(\mathscr{P}_{\mathrm{Syz}(M)}) = 0, \quad i=0,\dots,d.$$

But

$$(\mathscr{P}_L - \mathscr{P}_{M_{<d}} - \mathscr{P}_{\mathrm{Syz}(M)}) = (\langle L\rangle - \langle M_{<d}\rangle - \langle \mathrm{Syz}(M)\rangle)(1+n\lambda+\cdots)$$

hence

$$\mathrm{cf}_i(\langle L\rangle) - \mathrm{cf}_i(\langle M_{<d}\rangle) - \mathrm{cf}_i(\langle \mathrm{Syz}(M)\rangle) = 0, \quad i=0,\dots,d.$$

By the very definition of L we know that $\mathrm{cf}_d(\langle L\rangle)=0$. Therefore, to prove (iv) it suffices to prove (i), (ii) and (iii). Arguing as before, it suffices to prove the corresponding statements for the series instead of proving them for the numerators. Now the proof follows immediately from the definitions. □

Definition 19. We define

$$\mathrm{Discr}_d(M) := -\ \mathrm{cf}_d(\langle \mathrm{Syz}(M)_{<d}\rangle^-) - \mathrm{cf}_d(\langle M_{<d}\rangle^-).$$

From Theorem 18 we know that it is a nonnegative integer.

The theorem allows to improve the algorithm MinSyz by using $\mathrm{Discr}_d(M)$. Namely, when such a number becomes 0, all the remaining critical pairs of M in degree d must be of *Type* S_0, hence they *can be discarded.* The tables at the end of the paper show the excellent behavior of this tool.

6. Computing minimal free resolutions

Now we address the problem of computing minimal free resolutions of graded submodules of free graded modules over polynomial rings. Of course a possible solution to this problem is to iterate the algorithm MinSyz, which computes a minimal set of syzygies. However, to improve the efficiency we have devised a "reading pattern" strategy. Let us explain what it is.

We represent the resolution as a matrix, whose columns are the modules of syzygies (the rightmost is the starting module) and whose rows are the degrees. We call *cell*(i,d) the computational zone of the ith module of the syzygies in degree d and we partition each cell in the top part and the bottom part. The top part contains the minimal critical pairs and the bottom part the vectors. Before starting our computation we eliminate the cells, which are useless because of some extra knowledge, such as the Castelnuovo regularity. Then we start from the lowest degree and we proceed from left to right, computing first in the (i,d) cell, with lowest d and highest i.

Suppose we have completed the whole top part of *cell*$(i+1,d)$, i.e., we have processed all the critical pairs related to it. Then we enter *cell*(i,d), where we find critical pairs in the top part and vectors in the bottom part. Some of the critical pairs (*Type G*) produce new elements in the Gröbner basis of M. The others (*Type S*) produce syzygies, i.e., incoming vectors in the *cell*$(i+1,d)$. These vectors may reduce to 0 or produce new minimal generators of $\mathrm{Syz}(M)$. In the last case they possibly produce new critical pairs in the cells *cell*$(i+1,\delta)$, with $\delta>d$. Then *cell*$(i+1,d)$ is completely done and we enter *cell*$(i-1,d)$, if not already discarded because of some extra knowledge. As we have already pointed out, part of the *Type* S_0 critical pairs are discarded by means of Hilbert functions.

7. Examples and timing

We report the timing (in seconds) obtained with the examples described below. They have been computed with a Pentium 133 MHz with 32 MB of memory, running under Linux. The second column of Table 1 shows the performance of the algorithm itself. The third one, labeled HDriven, reports the performance when the use of the Hilbert functions was enabled; the last one, labeled Regularity, shows the timing when we allowed the a priori knowledge of the Castelnuovo regularity. The "equal signs" mean that in the corresponding cases, the algorithm itself plus the Hilbert functions took care of the regularity, so that there was no extra improvement.

Examples

```
-- Catalecticant
Use R::=Z/(32003)[z[0..3,0..3,0..3]];
A:=Mat[
  [z[3,0,0], z[2,1,0], z[2,0,1]],
  [z[2,1,0], z[1,2,0], z[1,1,1]],
  [z[2,0,1], z[1,1,1], z[1,0,2]],
  [z[1,2,0], z[0,3,0], z[0,2,1]],
  [z[1,1,1], z[0,2,1], z[0,1,2]],
  [z[1,0,2], z[0,1,2], z[0,0,3]] ];
I:=Ideal(Minors(2,A));
```

$$0 \longrightarrow R(-9) \longrightarrow R^{27}(-7) \longrightarrow R^{105}(-6) \longrightarrow R^{189}(-5) \longrightarrow R^{189}(-4) \longrightarrow R^{105}(-3) \longrightarrow R^{27}(-2)$$

```
-- 11 generic points in P^6
Use R ::= Z/(32003)[x[0..6]];
I := GenericPointsIdeal(11,6);
```

$$0 \longrightarrow R^4(-8) \longrightarrow R^{18}(-7) \longrightarrow R^5(-5) \oplus R^{25}(-6) \longrightarrow R^{45}(-4) \oplus R(-5) \longrightarrow R^{46}(-3) \longrightarrow R^{17}(-2)$$

```
-- 2x2 Minors of a generic 3x5 matrix
Use R ::= Z/(32003)[x[1..3,1..5]];
A := Mat[[x[I,J]| J In 1..5]| I In 1..3];
I := Ideal(Minors(2,A));
```

$$0 \longrightarrow R^6(-10) \longrightarrow R^{40}(-9) \longrightarrow R^{105}(-8) \longrightarrow R^{50}(-6) \oplus R^{120}(-7) \longrightarrow R^{168}(-5) \oplus R^{50}(-6) \longrightarrow R^{210}(-4) \longrightarrow R^{120}(-3) \longrightarrow R^{30}(-2)$$

```
-- Cyclic 5
Use R ::= Z/(32003)[abcdef];
I := Ideal(a + b + c + d + e,   ab + bc + cd + de  + ea,
abc + bcd + cde + dea + eab,   abcd + bcde + cdea + deab + eabc,
abcde - f^5);
```

$$0 \longrightarrow R(-15) \longrightarrow R(-10) \oplus R(-11) \oplus R(-12) \oplus R(-13) \oplus R(-14) \longrightarrow R(-6) \oplus R(-7) \oplus R^2(-8) \oplus R^2(-9) \oplus R^2(-10) \oplus R(-11) \oplus R(-12) \longrightarrow R(-3) \oplus R(-4) \oplus R^2(-5) \oplus R^2(-6) \oplus R^2(-7) \oplus R(-8) \oplus R(-9) \longrightarrow R(-1) \oplus R(-2) \oplus R(-3) \oplus R(-4) \oplus R(-5)$$

```
-- Five dense polys of deg 4
Use R ::= Z/(32003)[abcd];
I := Ideal([Randomized(DensePoly(4))| I In 1..5]);
```

$0\longrightarrow R^{20}(-11)\longrightarrow R^{46}(-10)\longrightarrow R^{10}(-8)\oplus R^{20}(-9)\longrightarrow R^{5}(-4)$

```
-----------------------------------------------------------------
-- Commute 3x3
Use R::=Z/(32003)[x[1..3,1..3]y[1..3,1..3]];
A := Mat[ [ x[I,J] | J In 1..3 ] | I In 1..3 ];
B := Mat[ [ y[I,J] | J In 1..3 ] | I In 1..3 ];
C := A*B-B*A;
I := Ideal(Entries(C));
```

$0\longrightarrow R^{4}(-9)\oplus R(-10)\longrightarrow R^{32}(-8)\longrightarrow R^{3}(-6)\oplus R^{58}(-7)\longrightarrow R^{32}(-5)\oplus R^{28}(-6)\longrightarrow R^{2}(-3)\oplus R^{31}(-4)\longrightarrow R^{8}(-2)$

```
-----------------------------------------------------------------
-- Cyclic 6
Use R ::= Z/(32003)[gabcdef];
I := Ideal(Ideal(a + b + c + d + e + f,
ab + bc + cd + de + ef + fa,
abc + bcd + cde + def + efa + fab,
abcd + bcde + cdef + defa + efab + fabc,
abcde + bcdef + cdefa + defab + efabc + fabcd,
g^6 + abcdef));
```

$0\longrightarrow R^{2}(-20)\longrightarrow R^{2}(-14)\oplus R^{10}(-19)\longrightarrow R^{10}(-13)\oplus R(-16)\oplus R^{4}(-17)\oplus R^{15}(-18)\longrightarrow R(-10)\oplus R^{4}(-11)\oplus R^{16}(-12)\oplus R(-13)\oplus R^{2}(-14)\oplus R^{3}(-15)\oplus R^{5}(-16)\oplus R^{9}(-17)\longrightarrow R(-6)\oplus R(-7)\oplus R^{2}(-8)\oplus R^{4}(-9)\oplus R^{6}(-10)\oplus R^{11}(-11)\oplus R^{2}(-12)\oplus R^{2}(-13)\oplus R^{2}(-14)\oplus R(-15)\oplus R^{2}(-16)\longrightarrow R(-3)\oplus R(-4)\oplus R^{2}(-5)\oplus R^{2}(-6)\oplus R^{3}(-7)\oplus R^{3}(-8)\oplus R^{2}(-9)\oplus R^{3}(-10)\oplus R(-11)\longrightarrow R(-1)\oplus R(-2)\oplus R(-3)\oplus R(-4)\oplus R(-5)\oplus R(-6)$

```
-----------------------------------------------------------------
-- Generic Entries
Use R ::= Z/(32003)[xyzt];
M := 3; N := 4;  P := NewMat(M,N);
For I := 1 To M Do
  For J := 1 To N Do;  P[I,J] := Randomized(DensePoly(2));
  End;
End;
I := Ideal(Minors(2,P));
```

$0\longrightarrow R^{17}(-8)\longrightarrow R^{48}(-7)\longrightarrow R^{16}(-5)\oplus R^{32}(-6)\longrightarrow R^{18}(-4)$

```
-----------------------------------------------------------------
```

In Table 2 we show some statistical data.

We observe that MP := number of minimal pairs, i.e., the pairs which survived after minimizing them; and GP := number of Gröbner pairs, i.e., the pairs which produced Gröbner (nonminimal) elements; MS := number of minimal *Type S* pairs, i.e., the pairs which produced syzygies. Of course MP = GP + MS; $\mathrm{MS}_{\min}$:= Number of minimal *Type* $S_{\min}$ pairs, i.e., pairs which produced necessary minimal ith-syzygies for every i; MS_0 := Number of minimal *Type* S_0 pairs, i.e., pairs, which produced useless syzygies. Of course $\mathrm{MS} = \mathrm{MS}_{\min} + \mathrm{MS}_0$; HKS_0 := Number of H-Killed *Type* S_0 pairs, i.e., pairs

Table 1
Timing

Name	Normal	HDriven	Regularity
Catalecticant	5.59 s	5.31 s	=, =
11 points in P^6	55.88 s	34.58 s	=, =
2×2 of 3×5	6.84 s	7.57 s	=, =
Cyclic 5	9.36 s	2.20 s	2.11 s
5 Dense	9.01 s	7.49 s	=, =
Commute 3×3	3.70 s	2.31 s	=, =
Cyclic 6	351.54 s	134.89 s	=, =
Generic entries	14.56 s	9.75 s	5.86 s

Table 2
Statistics

Name	MP	GP	MS	MS_{min}	MS_0	HKS_0	HdS_0
Catalecticant	902	111	791	616	175	165	10
11 points in P^6	256	45	211	144	67	62	5
2×2 of 3×5	1350	152	1198	869	329	181	148
Cyclic 5	182	41	141	26	115	67	48
5 Dense	204	52	152	96	56	33	23
Commute 3×3	376	69	307	191	116	56	60
Cyclic 6	374	61	313	137	176	82	94
Generic entries	185	39	146	113	33	29	4

killed by the use of the Hilbert functions, as described in Section 5; $HdS_0 :=$ Number of Hard-*Type* S_0 pairs, i.e., useless pairs, which were not detected by the Hilbert functions. Of course $MS_0 = HKS_0 + HdS_0$.

References

[1] W.W. Adams and P. Loustaunau, An Introduction to Gröbner Bases, Graduate Studies in Mathematics (AMS, Providence, RI, 1994).
[2] A.M. Bigatti, Computations of Hilbert–Poincaré Series, J. Pure Appl. Algebra, to appear.
[3] D. Bayer and M. Stillman, Macaulay (Version 3.0), A computer algebra system for algebraic geometry.
[4] M. Caboara and C. Traverso, The powerful Buchberger algorithm for modules, Preprint, 1996.
[5] A. Capani, G. Niesi and L. Robbiano: CoCoA3, a system for doing computations in commutative algebra, 1995. Available via anonymous ftp from lancelot.dima.unige.it.
[6] D. Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry, Springer Graduate Texts in Mathematics, Vol. 150 (Springer, Berlin, 1995).
[7] R. Gebauer and M. Möller, On an installation of Buchberger's algorithm, in: Robbiano, ed., Computational Aspects of Commutative Algebra (Academic Press, New York, 1989) 141–152.
[8] H. Grassmann, G.M. Greuel, B. Martin, W. Neumann, G. Pfister, W. Pohl, H. Schönemann and T. Siebert, Standard bases, syzygies and their implementation in singular, Preprint, 1994.
[9] A. Giovini and G. Niesi, CoCoA 1.5, A system for doing computations in commutative algebra, 1990.
[10] G.M. Greuel, G. Pfister and H. Schönemann, Singular, a computer algebra system for algebraic geometry and singularity theory, 1995. Available via anonymous ftp from helios. mathematik. uni-kl-de.
[11] D. Hilbert, Über die Theorie von algebraischen Formen, Math. Ann. 36 (1890) 473–534.

[12] R. La Scala, Un approccio computazionale alle risoluzioni libere minimali, Quaderno 1/96, Ph.D. Thesis, University of Bari, 1996.
[13] H.M. Möller and T. Mora, New constructive methods in classical ideal theory, J. Algebra 100 (1986) 138–178.
[14] L. Robbiano and G. Valla, Free resolutions for special tangent cones, Proc. Trento Conf., Lecture Notes in Pure and Applied Math., Vol. 84 (Marcel Dekker, New York, 1983).
[15] F.O. Schreyer, Die Berechnung von Syzygien mit dem verallgemeinerten Weierstrass'schen Divisionsatz, Diplomarbeit, Hamburg, 1980.
[16] D. Spear, A constructive approach to commutative ring theory, Proc. 1977 MACSYMA User's Conf., NASA CP-2012 (1977) 369–376.